Cauchy-Riemann Equations in Polar Form

Apart from the direct derivation given on page 35 and relying on chain rule, these equations can also be obtained more geometrically by equating single-directional derivative of a function at any point along a radial line and along a circle (see picture):

Derivative along radial line:

$$\frac{df}{dz} = \frac{\partial f}{e^{i\theta}\partial r} = e^{-i\theta}\frac{\partial f}{\partial r} = e^{-i\theta}(u_r + iv_r)$$

Derivative along a circle:

$$\frac{df}{dz} = \frac{\partial f}{ire^{i\theta}\partial\theta} = -\frac{ie^{-i\theta}}{r}\frac{\partial f}{\partial\theta} = -\frac{ie^{-i\theta}}{r}(u_\theta + iv_\theta) = \frac{e^{-i\theta}}{r}(v_\theta - iu_\theta)$$

Equate these two expressions to obtain the result:

$$\boxed{\begin{aligned} u_r &= \frac{1}{r}v_\theta \\ v_r &= -\frac{1}{r}v_r \end{aligned}}$$